Abstract. This paper concerns the Taylor-Aris dispersion of a dilute solute concentration immersed in a highly heterogeneous fluid flow having possibly sharp interfaces (discontinuities) in the diffusion coefficient and flow velocity. The focus is two-fold: (i) Calculation of the longitudinal effective dispersion coefficient, and (ii) Sample path analysis of the underlying stochastic process governing the motion of solute particles. Essential complete solutions are obtained for both problems.

1. History.
Dispersion of a solute concentration immersed in Poisseuille flow $U(y)$ directed along the horizontal axis of an infinite cylinder of radius $R$. The molecular diffusion is $D(y)$. No flux at boundary.

Find the asymptotic longitudinal dispersion $D$.

2. The general problem.
Concentration of solute: $c(x,t)$. Diffusion in an infinite cylinder $G$ of cross-section $G$ with diffusion tensor $D$ and flow velocity $U$: $G$ has smooth boundary and normal vector $n$, $U(y) = \text{vector field}$, and $D(x)$ is diagonal and uniformly positively definite.

Homogenization of the longitudinal process.
Cross sectional average of $c(x,t)$:
\[
C(x,t) = \int_G c(x,t) \, dx
\]
Homogenized concentration:
\[
\hat{C}(x,t) = \int_G C(\lambda x,t) \lambda \, dx
\]

Theorem (Generalized Taylor-Aris Formula)
\[
C(x,t) \text{ satisfies the homogenized equation}
\]
\[
\hat{C}(x,t) = \frac{\partial}{\partial t} \hat{C}(x,t) - \nabla \cdot (D \nabla \hat{C}(x,t))
\]

3. Layered medium.
\[
d = 2, \quad G = [0,1] \times [0,1]
\]

Sharp interfaces:
\[
a = 1, \ldots, L, L+1 = \ldots = L_M = \ldots = 1,\quad L_L < L_{L+1} < \ldots < L_M < L_M+1 = b
\]

4. Probabilistic formulation.
Underlying stochastic process:
\[
X_t = \int_0^t \mathcal{C}(\cdot,\cdot) + s \, dB_t
\]

5. Skew diffusion.
PDE for the concentration $c$:
\[
\partial_t c = \partial_j (D \partial_j c) - \nabla \cdot (D \nabla c) + \alpha \partial_j c
\]
Fundamental solution - transition probabilities for $\gamma(t)$:
\[
\gamma(t)_{a,b} = \int \gamma(t)_{a,b,c,d} \, d\mu(c,d)
\]

Some features of $\gamma(t)_{a,b}$:

\[
\gamma(t)_{a,b} \approx \text{Brownian motion on } \mathbb{R}^2 \text{ with diffusion coefficient } D = D(y)
\]

References